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Temperature field in non-Newtonian flow over a stretching plate with variable heat flux

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Abstract—This study deals with the temperature distribution in a visco-elastic fluid of Walters' liquid B model over a horizontal stretching plate. The velocity of the plate is proportional to the distance from the slit and the plate is subject to variable heat flux. The results are expressed in terms of Kummer's functions. Several closed-form solutions for specified conditions are presented. The effect of the visco-elastic parameter Ko^* and the heat flux parameter s on the temperature field is also studied. In addition, the solutions of a linearly stretching plate in a Newtonian flow with variable surface heat flux are also obtained. When $Ko^* = 0$ and $s = 0$, the solutions reduce to the published results.

INTRODUCTION

BOUNDARY-LAYER behaviour on a moving continuous solid surface is an important type of flow occurring in a number of engineering processes. An example of a moving continuous surface is a polymer sheet or filament extruded continuously from a die, or a long thread travelling between a feed roll and a wind-up roll.

Flow in the boundary layer on a continuous solid surface with constant speed was studied by Sakiadis [1]. Due to entrainment of ambient fluid, this situation represents a different class of boundary layer problem which has a solution substantially different from that of boundary layer flow over a semi-infinite flat plate. Erickson *et al.* [2] extended this problem to the case in which suction or blowing existed at the moving surface. Since polyester is a flexible material, the filament surface may stretch during the course of ejection and therefore the surface velocity deviates from being uniform. Crane [3] considered a moving strip the velocity of which is proportional to the distance from the slit. These types of flow usually occur in the drawing of plastic films and artificial fibres. The heat and mass transfer on a stretching sheet with suction or blowing was investigated by Gupta and Gupta [4]. They dealt with the isothermal moving plate and obtained the temperature and concentration distributions. Dutta *et al.* [5] analysed the temperature distribution in the flow over a stretching sheet with uniform heat flux. It is shown that the temperature at a point decreases with an increase in the Prandtl number.

More recently Siddappa and Abel [6] studied the non-Newtonian flow past a stretching plate and obtained the solution of the equation of motion. Grubka and Bobba [7] considered the heat transfer occurring on a continuous, linearly stretched surface with a power law surface temperature. In the present investigation, the heat transfer in a visco-elastic fluid of Walters' liquid B model over a stretching plate subject to power law flux has been studied. A series solution to the energy equation in terms of Kummer's functions is obtained. Several closed-form analytical solutions are also presented for special conditions.

In many practical fluids such as plastic films and artificial fibres, the hypothesis of a Newtonian fluid is obviously unsuitable. Therefore, the problem of determining the temperature field in a non-Newtonian fluid over a stretching surface does not seem to have received any attention. The present study is addressed to this problem.

ANALYSIS

Consider a steady visco-elastic two-dimensional flow past a horizontal stretching plate that issues from a thin slit at $x = 0$, $y = 0$, as in a polymer processing application (Fig. 1). It is assumed that the speed of a point on the plate is proportional to its distance from the slit, the boundary layer approximations are still applicable, and viscous dissipation is neglected in the energy equation.

The steady-state boundary layer equation governing the flow of visco-elastic fluid (Walters' liquid B) [6] is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - Ko^*\left\{u\frac{\partial^3 u}{\partial x\,\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y}\right\} \quad (1)$$

the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad (2)$$

and the energy equation is

NOMENCLATURE

f similarity solution of equations (1) and (2)
g similar dimensionless temperature function
k thermal conductivity
Ko^* visco-elastic parameter
P dimensionless parameter in equation (16)
Pr Prandtl number
q surface heat flux
s heat flux parameter
T temperature
u, v velocity component in x, y direction.

Greek symbols
α thermal diffusivity
η similarity variable
θ dimensionless temperature
ν kinematic viscosity of fluid
ψ stream function.

Superscript
′ derivative with respect to η.

Subscripts
y derivative with respect to y
w stretching plate conditions
∞ ambient conditions.

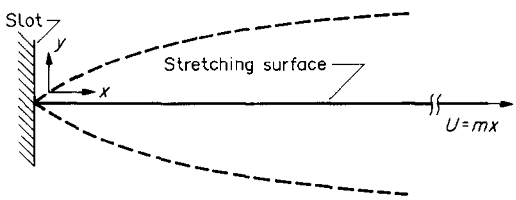

FIG. 1. Boundary layer on a stretching plate.

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \quad (3)$$

where (x, y) are the coordinates, (u, v) the velocity components in these directions, and Ko^* the visco-elastic parameter. The relevant boundary conditions are

$$u = mx,\quad v = 0,\quad -k\frac{\partial T}{\partial y} = q_w = Ax^s \quad \text{for } y = 0$$
$$u = 0,\quad v = -C,\quad u_y = 0,\quad T = T_\infty \quad \text{as } y \to \infty \quad (4)$$

where m and A are given constants, C is a positive constant to be determined and subscript y denotes differentiation with respect to y.

Since the fluid is incompressible, the momentum equation (1) and energy equation (3) can be solved consecutively. The solution to the momentum equation will be considered first. A stream function ψ defined by

$$u = \frac{\partial \psi}{\partial y},\quad v = -\frac{\partial \psi}{\partial x} \quad (5)$$

is introduced such that the continuity equation is identically satisfied.

A dimensionless stream function is given by

$$\psi = (mx/r)f(\eta),\quad \eta = ry. \quad (6)$$

Here r is a positive constant to be determined from equation (1), f is the dimensionless stream function and the similarity variable η depends on y only. Using equations (5) and (6), the velocity components become

$$u = mxf'(\eta),\quad v = -\frac{m}{r}[f(\eta)-f(0)] \quad (7)$$

where a prime denotes differentiation with respect to η. We can set $f(0) = 0$ in equation (7) without loss of generality so that

$$v = -(m/r)f(\eta). \quad (8)$$

Putting these values of u and v in equation (1) it becomes

$$(f')^2-ff'' = (\nu r^2/m)f'''-Ko^* r^2\{2f'f''' -ff^{iv}-(f'')^2\} \quad (9)$$

which is subject to the boundary conditions

$$f'(0) = 1 \qquad \text{for } \eta = 0$$
$$f'(\infty) = 0,\quad f''(\infty) = 0,\quad C = \frac{m}{r}f(\infty) \quad \text{as } \eta \to \infty. \quad (10)$$

In order to satisfy the above boundary conditions, Siddappa and Abel [6] have suggested to try a solution of the form

$$f'(\eta) = e^{-\eta},\quad f(\eta) = 1-e^{-\eta}. \quad (11)$$

Then equation (9) becomes

$$r = \sqrt{\left(\frac{1}{(\nu/m-Ko^*)}\right)} \quad (12)$$

hence the solution of equation (1) is obtained as

$$u = mx e^{-\eta},\quad v = -\frac{m}{r}(1-e^{-\eta}). \quad (13)$$

To solve the energy equation (3), the temperature distribution can be taken in the form of a similar solution as

$$T - T_\infty = \frac{Ax^s}{kr} g(\eta). \tag{14}$$

By substituting equations (13) and (14) into equation (3), one obtains

$$g'' + P(1 - e^{-\eta})g' - Ps\,e^{-\eta} g = 0 \tag{15}$$

where P is the modified Prandtl number in the visco-elastic fluid expressed as

$$P = (\nu/\alpha - m\,Ko^*/\alpha). \tag{16}$$

It should be noted that the dimensionless parameter P relates the relative magnitudes of diffusion of momentum and heat in the visco-elastic fluid. For a Newtonian fluid, i.e. $Ko^* = 0, P$ in equation (16) reduces to Pr. The boundary conditions for g are derived from equations (4) and (14) as

$$g'(0) = -1, \quad g(\infty) = 0. \tag{17}$$

Introducing a new variable $\xi = -P\,e^{-\eta}$ and substituting the solution for f into equation (15) gives

$$\frac{d^2 g}{d\xi^2} + (1 - P - \xi)\frac{dg}{d\xi} + sg = 0 \tag{18}$$

with the boundary conditions

$$\frac{dg(\xi = -P)}{d\xi} = -1/P, \quad g(\xi = 0) = 0. \tag{19}$$

The solution of equation (18) satisfying equation (19) in terms of Kummer's functions [8] is

$$g(\xi) = \frac{1}{P}\left(-\frac{\xi}{P}\right)^P \frac{M(P-s, P+1, \xi)}{M(P-s, P, P)} \tag{20}$$

where

$$M(a,b,z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{b_n}\frac{z^n}{n!}$$

$$a_n = a(a+1)(a+2), \ldots, (a+n-1)$$

$$b_n = b(b+1)(b+2), \ldots, (b+n-1). \tag{21}$$

Rewriting equation (20) in terms of η

$$g(\eta) = \frac{1}{P} e^{-P\eta} \frac{M(P-s, P+1, -P\,e^{-\eta})}{M(P-s, P, -P)}. \tag{22}$$

The wall temperature T_w is obtained from equation (14) as

$$T_w - T_\infty = \frac{Ax^s}{kr} g(0) \tag{23}$$

where

$$g(0) = \frac{1}{P}\frac{M(P-s, P+1, -P)}{M(P-s, P, -P)}. \tag{24}$$

It is worth pointing out that the dimensionless temperature distribution $\theta = (T - T_\infty)/(T_w - T_\infty)$ is equal to the ratio of $g(\eta)$ to $g(0)$. Several closed-form solutions are developed from equations (22) and (24) for specific values of s and P. Four such cases are reported in Table 1.

RESULTS AND DISCUSSION

Equations (22) and (24) were evaluated to determine the temperature field and the surface temperature as a function of P and s. It should be noted that the solutions of a continuous, stretching surface in a Newtonian fluid with variable surface heat flux are

$$g(\eta) = \frac{1}{Pr} e^{-Pr\eta} \frac{M(Pr-s, Pr+1, -Pr\,e^{-\eta})}{M(Pr-s, Pr, -Pr)}. \tag{25}$$

For a uniform heat flux in a Newtonian fluid, i.e. $s = 0$, equation (25) reduces to that reported by Dutta *et al.* [5].

As the numerical value of the fluid visco-elastic parameter Ko^* decreases, the dimensionless parameter P increases. Temperature fields were obtained for $P = 0.3, 0.5, 1, 3$, and 5 with s ranging between -2 and 2. The effect of the heat flux parameter s on θ is illustrated in Fig. 2 for $P = 0.7$. Figure 2 shows that the wall temperature gradient is negative for $s = 0$, 1, and 2. This implies that the heat flows from the continuous surface to the ambient. The magnitude of the temperature gradient increases with increasing s. When $s = -2$, the sign of the temperature gradient changes but the value of $g(0)$ is negative; and hence the heat flux at the surface flows into the fluid. It can

Table 1. Dimensionless temperature and wall temperature expressions for various P and S

S	P	θ	$g(0)$
P	S	$(e^{-P\eta})/P$	$1/P$
0	1	$e^{(1-\exp(-e^{-\eta}))}$	$e-1$
0	P	$e^P P^{-P}\gamma(P, P\,e^{-\eta})$	$e^P P^{-P}\gamma(P,P)$†
-2	P	$-\frac{1}{P}(1+P-P\,e^{-\eta})\exp(P(1-\eta-e^{-\eta}))$	$-1/P$

† γ incomplete Gamma function.

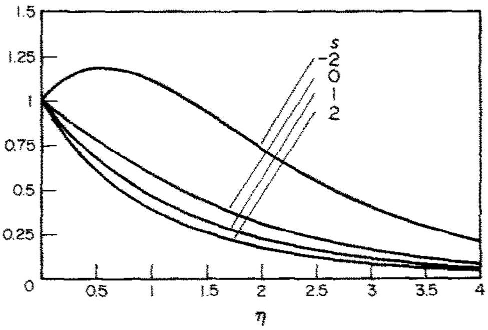

FIG. 2. Dimensionless temperature field for various s at $P = 0.7$.

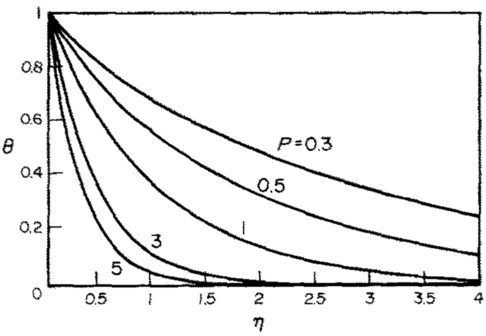

FIG. 3. Dimensionless temperature field for various P at $s = 1$.

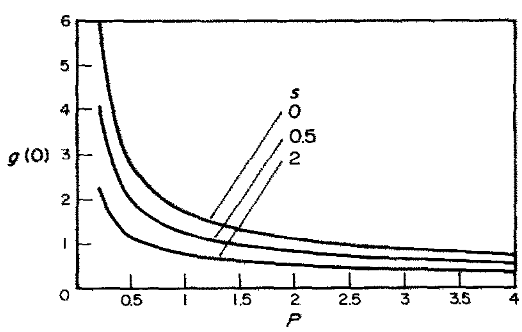

FIG. 4. Dimensionless surface temperature for various s.

be seen in Fig. 2 that a zero temperature gradient occurs at $s = -2$, where a minimum temperature exists in the thermal boundary layer. Therefore, the heat flows into the thermal boundary layer from both the ambient free stream and the stretching plate.

The dimensionless temperature field θ vs η for various P at $s = 1$ are plotted in Fig. 3. It is shown that the temperature at a point decreases with decreases in the visco-elastic parameter Ko^*. For a given s value, the larger the P, the smaller the thermal boundary layer thickness. The dimensionless surface temperature, $g(0)$, variation with P is given in Fig. 4. The surface temperature decreases rapidly as P increases from 0 to 1 and then slowly decreases with increases in P. Figure 4 also shows that the larger the heat flux parameter s, the smaller the surface temperature.

CONCLUSIONS

In this study, the heat transfer in a visco-elastic fluid of Walters' liquid B model over a linearly horizontal stretching plate with a power law heat flux has been solved in terms of Kummer's functions. Several closed-form solutions for specified conditions are presented.

The thermal boundary layer thickness decreases with a decrease in the visco-elastic parameter, Ko^*, i.e. an increase in the modified Prandtl number, P. Varying the heat flux parameter s affects the mechanism of heat transfer. The temperature field in the Newtonian fluid over a stretching plate with variable heat flux is also included in this work. When $P = Pr$, $s = 0$, the solutions reduce to the published results [5].

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CHAMP DE TEMPERATURE DANS UN ECOULEMENT NON-NEWTONIEN SUR UNE PLAQUE AVEC UN FLUX DE CHALEUR VARIABLE

Résumé—Cette étude concerne la distribution de température dans un fluide visco-élastique selon le modèle liquide B de Walter sur une plaque horizontale. La vitesse de la plaque est proportionnelle à la distance de la fente et la plaque est soumise à un flux variable. Les résultats sont exprimés en fonction des variables de Kummer. Quelques solutions analytiques pour des conditions spécifiques sont présentées. On étudie aussi l'effet du paramètre visco-élastique Ko^* et des paramètres de flux thermique sur le champ de température. De plus, les solutions sont obtenues pour une plaque dans un écoulement newtonien, avec un flux pariétal thermique variable. Lorsque $Ko^* = 0$ et $s = 0$, les solutions se réduisent aux résultats déjà publiés.

TEMPERATURFELD IN EINER NICHT-NEWTONSCHEN STRÖMUNG ÜBER EINE AUSGEDEHNTE PLATTE BEI VERSCHIEDENEN WÄRMESTROMDICHTEN

Zusammenfassung—Diese Studie beschäftigt sich mit der Temperaturverteilung in einem viskoelastischen Fluid (Flüssigkeitsmodell B nach Walter) über eine horizontal ausgedehnte Platte. Die Geschwindigkeit der Platte ist proportional zur Entfernung vom Spalt; der Platte werden verschiedene Wärmeströme aufgeprägt. Die Ergebnisse sind in Form von Kummer-Funktionen ausgedrückt. Für spezielle Bedingungen werden verschiedene geschlossene Lösungen angeboten. Die Auswirkung des viskoelastischen Parameters Ko^* und des Wärmestromparameters s auf das Temperaturfeld werden ebenso untersucht. Zusätzlich erhält man die Lösungen für eine linear ausgedehnte Platte in einer newtonschen Strömung mit verschiedenen Oberflächenwärmeströmen. Für $Ko^* = 0$ und $s = 0$ führen die obigen Lösungen auf die bekannten Ergebnisse.

ТЕМПЕРАТУРНОЕ ПОЛЕ ПРИ ОБТЕКАНИИ НЕНЬЮТОНОВСКОЙ ЖИДКОСТЬЮ ПЛАСТИНЫ, НАГРЕВАЕМОЙ ПЕРЕМЕННЫМ ТЕПЛОВЫМ ПОТОКОМ

Аннотация—Исследуется температурное поле в вязкоупругой жидкости Валтерса (В) вокруг горизонтальной формуемой пластины. Скорость пластины пропорциональна расстоянию от щели. Пластина нагревается переменным тепловым потоком. Результаты представлены в виде функций Куммера для конкретных условий. Приведены несколько решений в замкнутой форме и изучено также влияние на температурное поле параметра вязкоупругости Ko^* и параметра теплового потока s. Кроме того, получены решения для пластины, движущейся по линейному закону В ньютоновской жидкости при переменном тепловом потоке на поверхности. При $Ko^* = 0$ и $s = 0$ решения дают ранее известные результаты.